

02-20-08

Jeff Adams Seminar

Tits Group

References: Tits ^{original} paper from 1966, Kottwitz has a new way of looking at this.

let G be a reductive alg. group, split with split torus T .

$$W = N_G(T)/T = N/T.$$

$$1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1 \text{ exact.}$$

Does the sequence split? If not, understand N .

If ~~not~~ ~~split~~ doesn't split, find finite subgroup

$$\tilde{W} \subset N \text{ s.t. } 1 \rightarrow A \xrightarrow{\pi} \tilde{W} \rightarrow W \rightarrow 1$$

where $A := \ker(\tilde{W} \rightarrow W)$.

Ex: ① For $G = GL(n)$, sequence splits since $W = S_n \subset N_G(T)$.

② $SL(2)$, ~~is~~ $1 \rightarrow T \rightarrow N \rightarrow W \rightarrow 1$ doesn't split.

$$N = \left(\begin{smallmatrix} \mathbb{Z} & \\ & \mathbb{Z} \end{smallmatrix} \right) \cup \left(\begin{smallmatrix} & \omega \\ -\gamma_W & \end{smallmatrix} \right).$$

$$\begin{pmatrix} w \\ -y_w \end{pmatrix}^2 = -I, \quad \infty$$

$T \rightarrow N \rightarrow \text{bad}$

$$\text{order } 2 \leftarrow \begin{pmatrix} w \\ -y_w \end{pmatrix}$$

can't split.

$$\text{But } I \rightarrow \pm I \rightarrow \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \rightarrow W \rightarrow I$$

II/2 " II/2/2

Let $(G, B, T, \{X_\alpha\})$ splitting data.

So get $\{\varphi_\alpha : SL(2) \rightarrow G\}$ such that $d\varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = X_\alpha$

Define $\sigma_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. and $\text{Im}(\varphi_\alpha) = (\text{Cent}(\text{Ker}(\alpha))^\text{derived})$

for α simple.

~~Defn:~~ Note: $\sigma_\alpha^{-1} = m_\alpha = \alpha^\vee(-1) \in H$ max'l
Tors

Defn: $\tilde{W} = \langle \sigma_\alpha \rangle$ = group generated by σ_α .

(2)

\mathbb{G}_m , \mathbb{Z} onto map $\tilde{W} \xrightarrow{\sim} W$

What's the kernel?

hm: let $T_0 := \langle m_\alpha \rangle \subseteq T \cong (\mathbb{Z}/2\mathbb{Z})^k$, $k \leq \text{semisimple rank}$

Then \tilde{W} has generators and relations $\langle T_0, \{b_\alpha : \alpha \text{ simple}\} \rangle$

Relations: (i) $\sigma_\alpha^2 = m_\alpha \in T_0$

(ii) $\sigma_\alpha h \sigma_\alpha^{-1} = s_\alpha(h)$ for $h \in T_0$.

(iii) Braid Relations

Braid Relations: If α, β are roots, let

$m_{\alpha, \beta}$ = order of $s_\alpha s_\beta = 1, 2, 3, 4, 6$.

$$\underbrace{\sigma_\alpha \sigma_\beta}_{m_{\alpha, \beta}} = \underbrace{\sigma_\beta \sigma_\alpha}_{m_{\alpha, \beta}} = \underbrace{\sigma_\beta \sigma_\alpha}_{m_{\alpha, \beta}} = \underbrace{\sigma_\beta \sigma_\alpha}_{m_{\alpha, \beta}}$$

W has generators $\{s_\alpha\}$, relations braid relation and $s_\alpha^2 = 1$.

(3)

Note: If G is s.c., $H_0 \cong (\mathbb{Z}/\alpha\mathbb{Z})^{\text{rank}(G)}$

Now let $w \in W$. Get ! $\tilde{w} \in \tilde{W}$ s.t.

$$\begin{array}{ccccc} I & \rightarrow & H_0 & \rightarrow & \tilde{W} \rightarrow W \rightarrow I \\ & & & & \tilde{w} \leftarrow w \end{array} \quad \begin{matrix} \text{set} \\ \text{theoretic} \\ \text{section} \end{matrix}$$

The way you do it is by letting

$$w = s_{d_1} s_{d_2} \cdots s_{d_n}, \text{ send it to } \tilde{w} = \sigma_{d_1} \cdots \sigma_{d_n}$$

Lemma: This is well-defined.

Cor: $\varphi: W \rightarrow \tilde{W}$ is a pseudo homomorphism

$$\text{i.e. } \varphi(w_1 w_2) = \varphi(w_1) \varphi(w_2)$$

$$\text{provided } \cancel{\ell(w_1 w_2)} = \ell(w_1) + \ell(w_2)$$

~~(2)~~

~~(3)~~

Defn: Let B is the Braid group, i.e. Free group \mathcal{I}

modulo the Braid relation. \mathcal{I} = free group on s_α

Note: $W = \mathcal{I} / \{ \text{braid relations } s_\alpha^2 = 1 \}$

So

$$\mathcal{B} \rightarrow W \rightarrow I$$

Lemma: $\varphi: W \rightarrow Y$ pseudo homomorphism

$\Rightarrow \tilde{\varphi}: \mathcal{B} \rightarrow Y$ a homomorphism

Pf: Obvious



So, ~~(B)~~ - $\mathcal{B} \rightarrow \tilde{W} \rightarrow I$ onto, kernel K.

Start over: Suppose (W, S) is a Weyl group
set of simple reflections

let I = free group on $s \in S$. let i_s be the element
of I corresponding to $s \in S$.

Note: $S = \{ \text{set of simple reflections} \} \leftrightarrow \{\text{simple roots}\}$

$$I \rightarrow L \rightarrow I \rightarrow W \rightarrow I$$

$\begin{matrix} \uparrow \\ \text{brad.} \\ s^2=1 \end{matrix}$

(5)

Defn. $V :=$ maximal quotient of I :

$$I \rightarrow U \rightarrow V \rightarrow W \rightarrow I \quad \text{s.t.}$$

- ① U is abelian
- ② Braid relations hold in V

Lemma: V has generators $\{v_s : s \in S\}$ with relations
the braid relations and $[y v_s^2 y, v_{s'}^2] = 1$
 $\forall y \in V, s, s' \in S$ [commutator]

Note:

$$I \rightarrow \{\text{braid}\} \rightarrow I \rightarrow \mathcal{B} \rightarrow I$$

$$I \rightarrow \{\text{braid}, i_s^2\} \rightarrow I \rightarrow W \rightarrow I$$

$$I \rightarrow P \rightarrow \mathcal{B} \rightarrow W \rightarrow I$$

"

Kernel
(Pure Braid Group)

$$P = \{\text{braid}, i_s^2\} / \{\text{braid}\}$$

④

Lemma: $V = \mathbb{B} / \langle [\beta, \beta] \rangle$

$$1 \rightarrow \mathbb{B} / \langle [\beta, \beta] \rangle \rightarrow V \rightarrow W \rightarrow 1 \text{ exact.}$$

Theorem

~~Reflection~~: $\mathbb{B} / \langle [\beta, \beta] \rangle = \text{Free abelian group on all reflections in } W,$

$$\text{i.e. } \mathbb{Z} \{ \alpha : \alpha > 0 \}$$

Let $X_+ = \mathbb{Z} [\{ \alpha \mid \alpha > 0 \}]$

$$1 \rightarrow X_+ \rightarrow V \rightarrow W \rightarrow 1$$

Universal property of $V \Rightarrow$

$$V \rightarrow \tilde{W} \rightarrow 1 \text{ exact.}$$

$$V = \{ v_s : s \in S \} \cup \{ t_r : r \in R \}$$

"reflections"

s.t. ① v_s braid relation

$$\text{② } v_s^2 = t_s$$

$$\text{③ } v_s t_r v_s^{-1} = t_{\dots}$$

$$\text{④ } [t_s, t_r] = 1$$

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02-07-08

Tits Group part II

G split. let $\sigma_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $(G, H, B, \{\alpha\})$
splitting datum

 $\tilde{W} := \langle \sigma_\alpha \rangle$

Define $W^* := \langle X_\alpha \mid \alpha \text{ is simple root} \rangle$

Relations : let $y_\alpha = X_\alpha^2$.

- ① Braid relations on X_α
- ② $y_\alpha^2 = 1$
- ③ $[y_\alpha, y_\beta] = 1$ commutator
- ④ $X_\alpha Y_\beta X_\alpha^{-1} = Y_\beta Y_\alpha^{<\alpha, \beta>}$

Note! $\alpha, \beta \Rightarrow \langle y_\alpha \rangle \cong (\mathbb{Z}/2\mathbb{Z})^n$

Thm: $W^* \cong \tilde{W}$, if you assume G is simply connected

$$X_\alpha \mapsto \sigma_\alpha$$

$$Y_\alpha \mapsto m_\alpha = \alpha^\vee(-1)$$

(1)

Pf: Last time

$$W^* \rightarrow \tilde{W}.$$

Kernel: Well, any element in W^* can be

written in the form

$$\chi_{\alpha_1} \chi_{\alpha_2} \cdots \chi_{\alpha_n} y_{\beta_1} y_{\beta_2} \cdots y_{\beta_m}$$

So consider $\sigma_{\alpha_1} - \sigma_{\alpha_n} m_{\beta_1} - m_{\beta_m} = 1$. Then $\Rightarrow \sigma_{\alpha_1} - \sigma_{\alpha_n} = 1$.

Key: If G is simply connected, then $\frac{m_{\beta_1}}{\beta_1} \cdots \frac{m_{\beta_n}}{\beta_n} < m_{\beta_1} \cdots m_{\beta_n}$

\tilde{W} : Any element of \tilde{W} is of the form $\tilde{w}h$,

where $w \in W$ \tilde{w} is a canonical lift, via $\tilde{W} \rightarrow W$.

$$w = s_{\alpha_1} \cdots s_{\alpha_k} \text{ reduced } \rightarrow \sigma_{\alpha_1} \cdots \sigma_{\alpha_k} = \tilde{w}.$$

$$\text{So as a set, } \tilde{W} = W \times (\mathbb{Z}/2\mathbb{Z})^n$$

Recall from last time we had

$$I \rightarrow * \rightarrow I \rightarrow W \rightarrow I$$

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Free group on simple reflections.

$$I \rightarrow U \rightarrow V \rightarrow W \rightarrow I$$

V is the maximal quotient of I : such that Braid relations hold
and U is abelian

$$I \rightarrow P \rightarrow B \rightarrow W \rightarrow I$$

$$I \rightarrow \frac{P}{[P,P]} \rightarrow \frac{B}{[B,B]} \rightarrow W \rightarrow I$$

"

$$I \rightarrow P_{ab} \rightarrow \bigvee^V \rightarrow W \rightarrow I$$

$$I \rightarrow \text{Kernel} \rightarrow I \rightarrow B \rightarrow I$$

$i \mapsto b_i$

Claim: $P = \langle y b_{s_i}^2 y^{-1} : y \in B \rangle$

Now, let $c_s = v_s^2$. Then $P_{ab} = \langle y c_s y^{-1} : y \in V \rangle$

Lemma: In B , you have: \supseteq $w s_i w = s_i$, $s_i \in S$

Then $b_w b_{s_i} = b_{s_i} b_w$

where b_w = pseudo-homomorphism (w)

↑ from last time

$$w = s_1 \cdots s_n \quad \text{reduced}$$

$$b_w = b_1 \cdots b_n$$

Pf: Case 1: $\stackrel{\text{length function}}{l}(ws_i) = l(s_i w)$

$$l(w) + l(s_i) = l(w) + 1$$

So thus $b_w b_{s_i} = b_{ws_i}$, etc.

Case 2: $l(ws_i) = l(w) - 1$, $l(s_i w) = l(w) - 1$.

Defn: Suppose $r = wsw^{-1}$ in W . Define

$c_r := b_w b_s b_w^{-1}$ in \mathcal{B} . This is well-defined by last Lemma.

Cor: $P_{ab} = \langle c_r \text{ s.t. } r \in R \rangle$ where

$$R = \{wsrw^{-1} \mid s \in S\} \longleftrightarrow \{\text{positive roots } \alpha > 0\}.$$

Now, let $X_+ = \text{free abelian group on } R \cong \text{Free abelian group on positive roots.} \cong \mathbb{Z}^N$ where

$N = \# \text{ positive roots.}$ We write $[r] \in X_+$ where r is a reflection.

Defn: Consider $W \xrightarrow{\varphi} X_+ \times^{\text{obvious action}} W$

Recall if $w \in W$, define $R(w) = \{\alpha > 0 \mid w^{-1}\alpha < 0\}$

$$\text{so } l(w) = |R(w)|$$

$$R(w_1 w_2) = R(w_1) \cup w_1 R(w_2)$$

Then set $\varphi(w) = \left(\sum_{\alpha \in R(w)} [\alpha], w \right)$ $(\text{only when } l(w_1 w_2) = l(w_1) + l(w_2))$

⑤

Lemma:

φ is a pseudohomomorphism

$$\psi: \mathbb{B} \rightarrow X_+ \times W$$

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$$\rho: \mathbb{P} \rightarrow X_+$$

$$[\rho, \rho]_+ \rightarrow 1$$

$\mathbb{B}/$

$$[\rho, \rho] \rightarrow X_+ \times W$$

$$\boxed{V \rightarrow X_+ \times W}$$

$$\text{Thm: } 1 \rightarrow X_+ \rightarrow V \rightarrow W \rightarrow 1$$

Also, $\mathbb{P}_{ab} \cong X_+ \cong \mathbb{Z}[\alpha > 0] \cong \mathbb{Z}[c_r]$ exact.

(6)

Cor: All the c_r are independent in $X_+ \times W$

(6)

Back to \tilde{W} : \exists map $\beta \rightarrow N_G(T)$

$$b_1 \dots b_k \mapsto \sigma_1 \dots \sigma_k$$

$$\rho \longmapsto T$$

$$[\beta, \rho] \longmapsto 1$$

So get $V \xrightarrow{\gamma} N_G(T)$

Then define $\overline{W} := \text{im}(\gamma)$.

Then $\overline{W} \cong \tilde{W}$.

Recall: Assume G is of equal rank (outer automorphism = 1). G is a complex reductive group. $H \subset G$ a Cartan, $B \subset G$ a Borel. $W = N_G(H)/H$, finite group.

Define $X = \{x \in \text{Norm}_G(H) : x^2 \text{ is central}\}$ (He writes $x^2 \in \mathbb{Z}$, $\mathbb{Z} = \mathbb{Z}(G)$)
 / conjugation by H

(i.e. $x \sim h x h^{-1} : h \in H$)

X is finite if G is semisimple.

(If $G = n$ -torus, $X = n$ -torus)

Compute X . It comes down to the Tits Group.

Tits Group: We have SES $1 \rightarrow H \rightarrow N \rightarrow W \rightarrow 1$

$N = N_G(H)$. W is not a subgroup of G , but you'd like it to be. Want $W \hookrightarrow ?$ subgroup of G ?

Ex. 1) $G = GL(n)$, $W =$ permutation matrices

2) $G = GL(2)$, $W = I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3) If $SL(2)$, you're dead, since $W = \{I, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\} \not\subseteq \mathbb{Z}/4\mathbb{Z}$

Tits group is a subgroup of N mapping onto W .

define $s(w) = \sum_{\alpha_i} - s_{\alpha_i}$. This is independent of reduced expression. We'll write $w \mapsto \tilde{w}$, $s_\alpha \mapsto s_{\alpha'}$,

$$s_{\alpha_1} s_{\alpha_2} \mapsto s_{\alpha'_1} s_{\alpha'_2} \quad (\text{for } \alpha_1 \neq \alpha_2)$$

$$\begin{matrix} s_{\alpha_1}^2 & \xrightarrow{\cancel{\text{ }} \text{ }} & s_{\alpha'}^2 \\ " & & " \\ " & & \alpha' \end{matrix}$$

Going Back: We have $(G, B, H, \{X_\alpha\})$ - a splitting datum
 (You need this to split $\hookrightarrow G \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1$)

$\{X_\alpha\}$ are a set of root vectors for $\alpha \in \Pi$ simple.

Defn: If α is a simple root, you get $\sigma_\alpha \in N$

This comes from $SL(2)$ construction

i.e. Choose $\varphi: SL(2) \rightarrow G$ s.t.

$$\varphi(\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}) = \alpha(z) \quad \text{and}$$

$$d\varphi(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = X_\alpha.$$

Then define $\sigma_\alpha = \varphi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$.

Defn: $\tilde{W} = \langle \sigma_\alpha \rangle \subseteq N$

Thm: (Tits): ① $I \rightarrow H_0 \rightarrow \tilde{W} \rightarrow W \rightarrow I$

where

$$H_0 = \{\alpha^\vee(-1) \mid \alpha \text{ simple}\} \subseteq H$$

α^\vee is a cocharacter, i.e. $\alpha^\vee : \mathbb{C}^\times \rightarrow H$.

② Relations as before: Braid relation w on H_0 .

$$\sigma_\alpha^2 = \alpha^\vee(-1)$$

③ There's a canonical lift $s: W \rightarrow \tilde{W}$ as before.

Ex: If G is simply connected, $\tilde{W} = \tilde{W}_U$.

$R^\vee/2R^\vee = H_0$ = elements of order 2 in H .

In general, \tilde{W} is a quotient, so $\tilde{W}_U \rightarrow \tilde{W}$

Back to Combinatorial Construction: We have X .

You can break it up into pieces: $X[x] = \{x' \in X : x' \sim_{\tilde{G}} x\}$

Ex: $SL(2)$. $X = \{x \in M^2 : x \in \mathbb{Z}\}/\sim_H$
 $= \{I, -I, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}\}/\sim_H$

The other sheets in the normalizer are $\begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & z \\ -\frac{1}{z} & 0 \end{pmatrix}^2 = -I, \text{ so } \underline{\quad}$$

$$t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= \{ I, -I, t, -t, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}.$$

The conjugacy classes are $I, -I, \{t, -t, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

To compute $X[x]$: You can replace x by $w \cdot x = n x n^{-1}$ for $w \in W$. (Cross Action)

You can also do $x \rightarrow \sigma_\alpha x$ if α is an imaginary, x -noncompact root. x imaginary means x something.

$[x^2 \in \mathbb{Z}, \text{ so } \theta_x := \text{conjugation by } x \text{ is an involution, } H \rightarrow H]$
 $\text{So } \theta_x(\alpha) = \alpha \text{ is what it means to be imaginary}$
 $\theta_x(X) = -X_\alpha \leftarrow \text{non-compact}$

Final setting: Take the Tits group \tilde{W} . $\tilde{\tau}$ = automorphism of \tilde{W} (int(x) in our example). So you have $(\tilde{W}, \tilde{\tau})$. Inductively define a subset of \tilde{W} : start with 1.

So, to compute $X[x]$, start with x . You can conjugate by σ_α . So $\sigma_\alpha x \sigma_\alpha^{-1} = \sigma_\alpha(x(\sigma_\alpha)^{-1})x$. Then you can do $\sigma_\beta(x(\sigma_\alpha)^{-1})x, \dots, \tilde{w}x = y \tilde{\tau} y \sigma_\alpha y(\sigma_\alpha^{-1})y, \dots, \sigma_\beta y, \dots$
 provided that β is y noncompact

You end up getting $\{1, \sigma_\alpha x (\sigma_\alpha)^{-1}, \dots\} \cdot x$

Thm: $\{1, \sigma_\alpha x (\sigma_\alpha)^{-1}, \dots\} \cdot x \stackrel{\hookrightarrow \text{subset of } \tilde{W}}{=} X[x] \quad \text{i.e. } X[x] \stackrel{1-1}{\longleftrightarrow} \text{subset of } W$

Theorem: Irred. admissible rep's of G with infinitesimal character λ are parameterized by $X[x] \times {}^v X[y]$

Thm: $X[x] \stackrel{1-1}{\longleftrightarrow} K_x \backslash G/B \quad \text{by dual group}$